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COMMENT

A note on the invariants for the time-dependent oscillator

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Abstract. We give a procedure for the construction of equivalent families of invariants for Ermakov systems, starting directly from the equations of motion.

In a recent paper Colegrave and Abdalla (1983) discussed the attainment of two equivalent families of linear (and quadratic) invariants for an oscillator with variable mass or variable frequency. The problem of the time-dependent oscillator has been analysed in recent years by several authors in the classical (Lewis 1968, Eliezer and Gray 1976) and quantum context (Lewis and Riesenfeld 1969, Hartley and Ray 1981).

Colegrave and Abdalla (1983) started from the Hamiltonians

$$H_1 = p^2/2m_0 + \tfrac{1}{2}m_0\omega^2(t)q^2 \tag{1}$$

and

$$H_2 = p^2/2m(t) + \tfrac{1}{2}m(t)\omega_0^2 q^2, \tag{2}$$

and used the condition on the Poisson brackets for a function $I(t, q, p)$ to be an invariant

$$\partial I/\partial t + [I, H] = 0. \tag{3}$$

For the Hamiltonian H_1, for example, they found the two equivalent families of quadratic invariants

(i) $$I_1 = q^2/\sigma^2 + (\dot{\sigma}q - \sigma p/m_0)^2 \tag{4}$$

where

$$\ddot{\sigma} + \omega^2(t)\sigma = 1/\sigma^3. \tag{5}$$

This is the usual Ermakov-Lewis invariant (Lewis and Riesenfeld 1969).

(ii) $$I_2 = p^2/\rho^2 + \{m_0\rho q + [\dot{\rho}/\omega^2(t)p\}^2 \tag{6}$$

where

$$\ddot{\rho} - 2\dot{\omega}\dot{\rho}/\omega + \omega^2(t)\rho = \omega^4(t)/\rho^3. \tag{7}$$

For the Hamiltonian H_2 they get similar results.

In this comment we show that the search for equivalent families of invariants of this kind (the Ermakov invariants) can be generalised and a great number of invariant forms can be obtained if we start from the Newtonian equations of motion rather than from the Hamiltonian description (Moreira 1983).

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Consider the equation of motion for the one-dimensional motion of a particle

$$\ddot{x} = f(t, x, \dot{x}) \tag{8}$$

and suppose that the invariant has the form

$$I = I(x, \dot{x}, \rho, \dot{\rho}, t) \tag{9}$$

where ρ satisfies the auxiliary equation

$$\ddot{\rho} = g(t, \rho, \dot{\rho}). \tag{10}$$

If I is to be an invariant it must satisfy

$$\partial I/\partial t + \dot{x}\, \partial I/\partial x + \dot{\rho}\, \partial I/\partial \rho + f\, \partial I/\partial \dot{x} + g\, \partial I/\partial \dot{\rho} = 0. \tag{11}$$

Now, we take the equation for the time-dependent oscillator

$$f = -k(t)x \tag{12}$$

and suppose that I has a simple dependence on $\dot{x}$ and x.

Linear invariant

If I has the form

$$I = f_1(\rho, \dot{\rho}, t)\dot{x} + f_2(\rho, \dot{\rho}, t)x \tag{13}$$

the relation (11) leads to

$$g(t, \rho, \dot{\rho}) = \frac{-[\dot{x}(\dot{\rho}\, \partial f_1/\partial \rho + \partial f_1/\partial t + f_2) + x(\dot{\rho}\, \partial f_2/\partial \rho + \partial f_2/\partial t - kf_1)]}{(\partial f_1/\partial \dot{\rho})\dot{x} + (\partial f_2/\partial \dot{\rho})x}. \tag{14}$$

Our imposition that g does not have an explicit dependence on $\dot{x}$ and x conduces to the following condition for f_1 and f_2:

$$(\dot{\rho}\, \partial f_1/\partial \rho + \partial f_1/\partial t + f_2)(\partial f_2/\partial \dot{\rho}) = (\partial f_1/\partial \dot{\rho})(\dot{\rho}\, \partial f_2/\partial \rho + \partial f_2/\partial t - kf_1). \tag{15}$$

We take some illustrative cases

(i) Making $f_1 = f_1(\rho)$, $f_2 = f_2(\rho, \dot{\rho})$ we get from (13) and (14)

$$I = -\dot{\rho}x\, \mathrm{d}f_1/\mathrm{d}\rho + f_1\dot{x} \tag{16}$$

with the auxiliary equation

$$\ddot{\rho} = -\frac{kf_1}{(\mathrm{d}f_1/\mathrm{d}\rho)} - \dot{\rho}^2 \frac{(\mathrm{d}^2 f_1/\mathrm{d}\rho^2)}{(\mathrm{d}f_1/\mathrm{d}\rho)}. \tag{17}$$

The case where $f_1 = \rho$ leads to the invariant $J^{(q)}$ (Colegrave and Abdalla 1983)

$$I_3 = J^{(q)} = \rho\dot{x} - \dot{\rho}x$$

and

$$\ddot{\rho} + k\rho = 0.$$

(ii) If we make $f_2 = f_2(\rho)$ we get

$$I = f_2 x + (\dot{\rho}/k)(\mathrm{d}f_2/\mathrm{d}\rho) \tag{18}$$

where ρ satisfies the equation

$$\ddot{\rho} = -\dot{\rho}^2 \frac{(\mathrm{d}^2 f_2/\mathrm{d}\rho^2)}{\mathrm{d}f_2/d\rho} + \frac{\dot{\rho}\dot{k}}{k} - kf_2. \tag{19}$$

If $f_2 = \rho$, we obtain the invariant $J^{(\rho)}$ (Colegrave and Abdalla 1983)

$$I_4 = J^{(\rho)} = \rho x + \dot{\rho}\dot{x}/k$$

and

$$\ddot{\rho} - \dot{k}\dot{\rho}/k + k\rho = 0.$$

The equivalence of I_3 and I_4 was shown by Colegrave and Abdalla (1983).

Quadratic invariant

A similar analysis for the quadratic invariant

$$I = f_3(\rho, \dot{\rho}, t)\dot{x}^2 + f_4(\rho, \dot{\rho}, t)x\dot{x} + f_5(\rho, \dot{\rho}, t)x^2 \tag{20}$$

leads to the following equation for ρ

$$\ddot{\rho} = -\frac{\lambda\dot{x}^2 + mx\dot{x} + nx^2}{a\dot{x}^2 + bx\dot{x} + cx^2} \tag{21}$$

where

$$\begin{aligned} \lambda &= \partial f_3/\partial t + \dot{\rho}\,\partial f_3/\partial\rho + f_4, & a &= \partial f_3/\partial\dot{\rho} \\ m &= \partial f_4/\partial t + \dot{\rho}\,\partial f_4/\partial\rho + 2f_5 - 2kf_3, & b &= \partial f_4/\partial\dot{\rho} \\ n &= \partial f_5/\partial t + \dot{\rho}\,\partial f_5/\partial\rho - kf_4, & c &= \partial f_5/\partial\dot{\rho}, \end{aligned} \tag{22}$$

under the conditions:

$$\lambda/a = m/b = n/c. \tag{23}$$

(i) Making $f_3 = f_3(\rho)$, we get $f_4 = -\dot{\rho}\,\mathrm{d}f_3/\mathrm{d}\rho$ and f_5 satisfying the equation

$$\partial f_5/\partial\dot{\rho}(-\dot{\rho}^2\,\mathrm{d}^2 f_3/\mathrm{d}\rho^2 + 2f_5 - 2kf_3) + [(\partial f_5/\partial\rho)\dot{\rho} + \partial f_5/\partial t]\,\mathrm{d}f_3/\mathrm{d}\rho + k\dot{\rho}(\mathrm{d}f_3/\mathrm{d}\rho)^2 = 0. \tag{24}$$

The Ermakov invariant I_1 is a particular case ($f_3 = \rho^2$).

(ii) If we choose $f_5 = f_5(\rho)$, f_4 and f_3 are given by $f_4 = (\dot{\rho}/k)\,\mathrm{d}f_5/\mathrm{d}\rho$

$$\begin{aligned} &(\partial f_3/\partial\dot{\rho})(\dot{\rho}^2\,\mathrm{d}^2 f_5/\mathrm{d}\rho^2 - \dot{\rho}(\dot{k}/k)\,\mathrm{d}f_5/\mathrm{d}\rho + 2kf_5 - 2k^2 f_3) \\ &\quad - \mathrm{d}f_5/\mathrm{d}\rho(\partial f_3/\partial t + \dot{\rho}\,\partial f_3/\partial\rho) - (\dot{\rho}/k)(\mathrm{d}f_5/\mathrm{d}\rho)^2 = 0. \end{aligned} \tag{25}$$

If $f_5 = \rho^2$ we get the invariant I_2 and its auxiliary equation.

We give another example of the application of the same procedure: starting from the equation of motion

$$\ddot{x} = -k_1(t)x + k_2(t)/x^2,$$

a quadratic invariant is obtained

$$I = \rho^2\dot{x}^2/2 - \rho\dot{\rho}x\dot{x} + D\rho/2x^2 + x^2\dot{\rho}^2/2 + Cx^2/2\rho^2$$

with

$$\ddot{\rho} + k_1(t)\rho = C/\rho^3$$

and $k_2(t) = D\rho^{-1}$; D and C are constants.

The approach considered here leads to the Ermakov invariants of type (9) and to the auxiliary equations (10) for an explicitly time-dependent one-dimensional equation of motion. It can be applied for the case where equations (8) and (10) are coupled and can be extended for multi-dimensional non-Hamiltonian systems. We see (from (21) and (23), for example) that there is, in general, no limit to the number of these invariant forms. This great arbitrariness is introduced because we give one more dimension (represented by ρ) to the usual space (in our discussion, one-dimensional space), and the diverse invariants obtained from (21) and (23) correspond to different choices of coordinate systems. The relations (23) are just the compatibility conditions for the two equations (10) and (12) to admit an invariant with a specified form in this space. (When we search these invariants from the Noether or Lie symmetries the auxiliary equation is the condition for a point transformation to be a symmetry transformation (Lutzky 1978, Moreira 1983). Of course, the invariants so obtained will be, in general, functionally dependent (see Colegrave and Abdalla 1983). However, they can be useful in finding the solution of an equation starting from a known solution of the auxiliary equation (Ray 1981, Moreira 1983), and they can be employed for the quantisation of time-dependent potential systems (Lewis and Riesenfeld 1976, Hartley and Ray 1981).

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